PRINCIPLES OF ANALYSIS
LECTURE 8 - SEQUENCES

PAUL L. BAILEY

1. Review

We described how the natural numbers can be build from axioms of set theory; how to construct the integers from the natural numbers, and how to construct the rationals from the integers.

We developed the real numbers as the set of cuts in the rational number line. This set supports addition, multiplication, and an ordering satisfying these properties:

(F1) $(a + b) + c = a + (b + c)$;
(F2) $a + 0 = a$;
(F3) $a + (-a) = 0$;
(F4) $a + b = b + a$;
(F5) $(ab)c = a(bc)$;
(F6) $a \cdot 1 = a$;
(F7) $a \cdot a^{-1} = 1$ for $a \neq 0$;
(F8) $ab = ba$;
(F9) $(a + b)c = ac + bc$;
(O1) $a \leq a$;
(O2) $a \leq b$ and $b \leq a$ implies $a = b$;
(O3) $a \leq b$ and $b \leq c$ implies $a \leq c$;
(O4) $a \leq b$ or $b \leq a$;
(O5) $a \leq b$ implies $a + c \leq b + c$;
(O6) $a \leq b$ implies $ac \leq bc$ for $c \geq 0$.
(CM) every set of real numbers bounded above has a least upper bound.

Property **(CM)** is equivalent to the lack of gaps in the real number line; this lack of gaps was proved using the Cantor-Dedekind Theorem. The Schroder-Bernstein theorem helped show that there is a linear order on the cardinal numbers. It is the lack of gaps which insures that base β expansions produce real numbers, which leads to the proof the $|\mathbb{Q}| < |\mathbb{R}|$.

Exercise 1. Recommended practice exercises from the book:
Chapter 0 exercises 10,13,14,21,32,36,38,40

Date: September 15, 2003.

2. Triangle Inequality

Let $x \in \mathbb{R}$, and define the *absolute value* of x, denoted by $|x|$, by

$$|x| = \begin{cases} x & \text{if } x \geq 0; \\ -x & \text{if } x < 0. \end{cases}$$

Clearly $-|x| \leq x \leq |x|$ for all $x \in \mathbb{R}$. We think of this as the distance between x and 0. Moreover, we think of $|x - a|$ as the distance between x and another real number a.

Proposition 1. *Let $a, b \in \mathbb{R}$. If $a \leq b$, then $-b \leq -a$.*

Proof. This uses property **(O5)**. Take $a \leq b$ and add $-b$ to both sides to get $a - b \leq 0$. Now add $-a$ to both sides to get $-b \leq -a$. $\square$

Proposition 2 (Triangle Inequality)**.** *Let $a, b \in \mathbb{R}$. Then $|a + b| \leq |a| + |b|$.*

Proof. We have $-|a| \leq a \leq |a|$ and $-|b| \leq b \leq |b|$. Repeated application of property **(O6)** yield

$$-(|a| + |b|) \leq a + b \leq |a| + |b|.$$

Multiply both sides of the left inequality by -1 to obtain $-(a + b) \leq |a| + |b|$. Now $|a + b|$ is either $a + b$ or $-(a + b)$, and in either case, we see that $|a| + |b|$ is greater than it. $\square$

3. Sequences

Let A be a set. A *sequence* in A is a function $a : \mathbb{Z}^+ \to A$. We write a_n to mean $a(n)$, and we write $\{a_n\}_{n=1}^\infty$, or simply $\{a_n\}$, to denote the function a. We will primarily be interested in sequences of real numbers, that is, sequences in $\mathbb{R}$.

4. Limit Points of Sequences

Let $\{a_n\}_{n=1}^\infty$ be a sequence of real numbers and let $L \in \mathbb{R}$. We say that L is a *limit point* of $\{a_n\}_{n=1}^\infty$ if

$$\forall \epsilon > 0 \; \exists N \in \mathbb{Z}^+ \;\ni\; n \geq N \;\Rightarrow\; |a_n - L| < \epsilon.$$

In this case, we say that $\{a_n\}_{n=1}^\infty$ *converges* to L.

Proposition 3. *Let $\{a_n\}_{n=1}^\infty$ be a sequence in $\mathbb{R}$ and let $L_1, L_2 \in \mathbb{R}$. If $\{a_n\}_{n=1}^\infty$ converges to L_1 and to L_2, then $L_1 = L_2$.*

Proof. Suppose not, and set $d = |L_1 - L_2|$; then d is positive. Let $\epsilon = \frac{d}{4}$. Then by definition of limit, there exist positive integers N_1 and N_2 such that $n \geq N_1$ implies that $|a_n - L_1| < \epsilon$, and $n \geq N_2$ implies that $|a_n - L_2| < \epsilon$.

Let $N = \max\{N_1, N_2\}$. Then for $n \geq N$,

$$\begin{aligned}
d &= |L_1 - L_2| \\
&= |L_1 - a_n + a_n - L_2| \\
&= |L_1 - a_n| + |a_n - L_2| \qquad \text{by the Triangle Inequality} \\
&= |a_n - L_1| + |a_n - L_2| \\
&\leq \epsilon + \epsilon \\
&= \frac{d}{2}.
\end{aligned}$$

This is a contradiction; thus $L_1 = L_2$. □

Thus limits are unique when they exist, justifying the article *the* limit instead of "a limit point". We write $L = \lim_{n\to\infty} a_n$ to say that $\{a_n\}_{n=1}^\infty$ converges to L.

If a sequence has a limit, we say that it is *convergent*; otherwise it is *divergent*.

Example 1. Show that $\lim_{n\to\infty} \frac{1}{n} = 0$.

Proof. Let $\epsilon > 0$. By the Archimedean Principle, there exists $N \in \mathbb{N}$ such that $N > \frac{1}{\epsilon}$. This gives $\frac{1}{N} < \epsilon$. Note that if $n \geq N$, then $1 \geq \frac{N}{n}$, and $\frac{1}{N} \geq \frac{1}{n}$. Thus for $n \geq N$ we have

$$|\frac{1}{n} - 0| = \frac{1}{n} \leq \frac{1}{N} < \epsilon.$$

This proves that $\lim_{n\to\infty} \frac{1}{n} = 0$. □

Let $\{a_n\}_{n=1}^\infty$ be a sequence of real numbers. The *image* of $\{a_n\}_{n=1}^\infty$ is the image of the sequence as a function, that is, it is the set

$$\{a_n \mid n \in \mathbb{Z}^+\}.$$

Note that there is much more information in a sequence than in its image; for example, the sequences $\{1 + (-1)^n\}_{n=1}^\infty$ and $\{0, 2, 0, 0, 2, 0, 0, 0, 2, 0, 0, 0, 0, 2, \dots\}$ have the same image; the common image is $\{0, 2\}$, a set containing two elements.

5. Bounded Sequences

We say that $\{a_n\}_{n=1}^{\infty}$ is *bounded above* if there exists $a \in \mathbb{R}$ such that $a \geq a_n$ for every $n \in \mathbb{Z}^+$.

We say that $\{a_n\}_{n=1}^{\infty}$ is *bounded below* if there exists $b \in \mathbb{R}$ such that $b \leq a_n$ for every $n \in \mathbb{Z}^+$.

We say that $\{a_n\}_{n=1}^{\infty}$ is *bounded* if it is both bounded above and bounded below. Equivalently, $\{a_n\}_{n=1}^{\infty}$ is bounded if there exists $M > 0$ such that $a_n \in [-M, M]$ for every $n \in \mathbb{Z}^+$.

Proposition 4. *Every convergent sequence is bounded.*

Proof. Let $\{a_n\}_{n=1}^{\infty}$ be a convergent sequence with limit L. Let N be so large that for $n \geq N$ we have $|a_n - L| < 1$. And $|L|$ to both sides of this inequality and apply the triangle inequality to get, for every $n \geq N$,

$$|a_n| \leq |a_n - L| + |L| < 1 + |L|.$$

There are only finitely many terms of the sequence between a_1 and a_{N-1}; set

$$M = \max\{|a_1|, |a_2|, \dots, |a_{N-1}|, 1 + |L|\}.$$

Then $M \geq a_n$ for every $n \in \mathbb{Z}^+$, so $\{a_n\}_{n=1}^{\infty}$ is bounded. □

Department of Mathematics and CSCI, Southern Arkansas University
E-mail address: plbailey@saumag.edu